Mechanics of highly deformed elastic shells

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1. Introduction

The response of naturally curved elastic shells when they are highly deformed into the (geometrical) nonlinear regime is explored by focusing on two common shell configurations: cylindrical shells and spherical shells. In addition to thin shells which are normally considered for conventional materials, we have also probed the behavior and mechanisms of some relatively thick shells, in view of the current interest in biological and small-scale structures. The approach here is employment of continuum-based computational models for solving shells governed by linear elasticity and fully nonlinear geometry using ABAQUS. Our material choice is restricted to that of an isotropic linear elastic material. Moreover, qualitative experiments have also been carried out on hemispherical shells to unravel some of the physical mechanisms of their response under indentation.

Shell structures have been widely used in pipelines, aerospace and marine structures, automotive industry, large dams, shell roofs, liquid-retaining structures and cooling towers [1]. Recent advancements in micro-electromechanical systems and nanotechnology have opened new avenues for applications of shells at much smaller scales. Examples are many and vary from carbon nanotubes and nanometer-sized buckyballs to microcapsules for drug delivery, colloidal armors, flexible electronics, tissue engineering and regenerative medicine [2–5]. Shell structures are also ubiquitous in nature and arise at a range of length scales from the earth’s crust to microtubules and biomembranes, as well as in plants [6–8]. The current interest in understanding the behavior of living cells and subcellular components has further accentuated the importance of studying the behavior of shells. For example, much attention has been directed recently towards understanding the behavior of microtubules, which are often highly curved and buckled because of the state of stress in the cytoplasm. These studies have direct implications in understanding the physiological forces applied to microtubules, their mechanical coupling with the cytoskeleton and their role in altering cell mechanics and function [9–12]. Other examples are the mechanics of the cell membrane [13–17], the nuclear envelope [18,19] and even nanometer-sized viruses and retrovirus particles [20,21].

Despite their significance, many phenomenological aspects of the behavior of naturally curved shells are still ambiguous and pose fundamental challenges for applications of mechanics in new areas such as nanostructures and biology. Many of the nonlinear shell studies conducted in the past have been motivated by failure concerns related to conventional shell structures, and for that reason no considerable effort has been made to probe the response of these structures deep into the nonlinear regime. Emergence of novel applications at micron and submicron scales, where the material failure becomes less influential, motivates investigating the response of these structures deeply into the nonlinear regime.

The highly nonlinear behavior of elastic shells is mainly governed by inextensible or almost inextensible deformations, which are energetically preferred by the shell [22–26]. In large deformations, this leads to the appearance of structural features
such as dimensionless developable cone and curvature condensates and almost inextensible one-dimensional ridges [27–29]. These features are ubiquitous both in nature and in technology, as well as in everyday human life. Examples are kinking of a straw and indentation of a plastic bottle using a sharp pen, as well as the crushed coke-can and dried resin. Fig. 1 provides several examples of such phenomenon in shells over a range of length scales. The interplay of physics and geometry, which indeed leads to the appearance of the localized features shown in Fig. 1, can even play a critical role at the early stage of shell response. An example is the persistence of a pinch in a circular pipe which manifests itself at the earliest stage of deformation due to the dominant role of nearly inextensible deformations [30].

In this study, first, we study the mechanics of elastic spherical caps (i.e. segment of a spherical shell) under both point-like and flat plate indentation. Then, the response of elastic circular and oval cylindrical shells under pure bending is investigated when they are deformed deep into the nonlinear regime. Our study complements a wide range of previous scaling approaches based on continuum elasticity [31–33]; and continuum-based and molecular dynamics simulations for understanding the response of highly deformed plates and shells [34,35,74]. The detailed numerical computations enable a high-fidelity exploration of highly deformed states of elastic shells.

### 2. Indentation of elastic spherical caps

In this section, we study the deformation and mechanics of spherical caps clamped along the edge with radius $R$, thickness $t$, and center angle $\alpha$, under both flat plate and point indentations—see Fig. 2A. First, the response of these structures under rigid flat indentation is analyzed, Fig. 2. In our numerical simulations, free sliding has been assumed between the flat plate and the spherical cap. The computations were carried out with the following material parameter values: Young’s modulus $E = 100$ GPa and Poisson ratio $\nu = 0.3$. Four-node shell elements with reduced integration were employed in all calculations. No initial geometric or material imperfection was included in the computational models. To follow the post-buckling response of the structure, a stabilizing mechanism based on automatic addition of volume-proportional damping was employed. For each set of calculations, the damping value was decreased systematically to assure that the response is insensitive to this change.

With $Z$ as the deflection at the center, at the early stage of indentation, $Z < t$, the spherical cap flattens with reaction force, $F \propto Z^2$. As the indentation increases, a contact disc forms with a diameter that can be estimated from simple geometrical relation such as $2r/R = \sqrt{2Z}$, where $Z = Z/R$ and $r$ denotes the disc radius. For this deformation mode, a simple scaling law based on the balance of stretching and bending energy in the spherical shell
yields $F \approx Z^{3/2}$ [31]. Further increase in the indentation causes the central portion of the flattened area to lift off the rigid surface while the deformation stays axisymmetric, Fig. 2B. This transition, which is accompanied by a sudden decrease in the load-carrying capacity of the shell, leads to a dimple with a localized region of deformation along a circular ridge, Fig. 2B and C. Inside this ridge, the deformed shell has a form that is a mirror image of its original shape, sometimes referred to as mirror-buckling mode [36,37]. A similar response can also be achieved by point indentation of a spherical cap [24,26,38,39]. As the indentation is increased further, the symmetry breaks along this ridge in the form of vertices. These vertices are linked to each other by approximately straight ridges, Fig. 2D and E. During this transition, the shell retains its load-carrying capacity as the asymmetric deformation mode develops. Fig. 3 displays the deformation map for clamped elastic spherical caps subjected to rigid flat indentation, which was constructed by systematically varying the thickness of the shell over two orders of magnitude, $0.0005 < t/R < 0.1$. For the wide range of shell thickness considered, the critical indentation associated with separation of the central region of the shell from the rigid indenter scales approximately linearly with the shell thickness, as also does the critical indentation associated with the transition from the axisymmetric deformation mode-to-asymmetric mode. For thin shells with $t/R < 0.0015$ in Fig. 3, transition from the symmetric deformation to asymmetric pattern results in the formation of four vertices, while for thicker shells, localization of deformation involves five vertices attached by five approximately straight ridges. A limited set of numerical simulations was carried out to investigate the role of shell center angle, $\alpha$, on the indentation response with variations between $120^\circ$ and $240^\circ$. The deformation is always limited to small regions of the shell, far from the clamped boundary. Since the calculations were limited to relatively deep shells, no qualitative difference was observed by varying the center angle, although for shells with larger center angle, the asymmetric deformation mode with four vertices appears in somewhat thicker shells.

A complementary set of qualitative experiments and numerical simulations was performed to study the mechanics of elastic spherical caps under point indentation. A qualitative experiment on point-like indentation of a hemispherical shell was carried out.

Fig. 2. (color online) Indentation of a spherical cap with a flat rigid plate. (A) Schematic of a clamped spherical cap subject to rigid flat plate indentation. (B) Two deformation modes of the spherical cap after the initial global deformation. (C) Force-indentation response of a spherical cap with $t/R = 0.01$ and $\alpha = 120^\circ$, under flat rigid plate indentation. The transition from the axisymmetric deformation to the asymmetric deformation with five vertices occurs at $Z = Z/R \approx 0.132$. (D) Distribution of the elastic energy density in an elastic spherical cap with $t/R = 0.01$ and $\alpha = 120^\circ$ at two different normalized indentations. (E) Localization of deformation in a plastic spherical cap with $t \approx 0.35$ mm, $t/R \approx 100$ and $\alpha = 180^\circ$, indented by a flat thick glass plate. The plastic was covered by a thin layer of oil to minimize friction and also enhance the clarity of the picture.
As the indentation displacement is increased, the shell first deforms to form a circular dimple, similar to the indentation by a rigid flat plate. Under further indentation, this deformation mode loses its symmetry to a polygonal shape with three vertices attached to each other by approximately straight ridges as well as to the indentation point. The transition from the axisymmetric mode-to-asymmetric mode is not accompanied by reduction in the load-carrying capacity of the shell [39]. Further indentation leads to formation of additional vertices and ridges, Fig. 4A. A simple geometrical argument for the position of the vertices can be obtained by equating the length of the ridge that connects the vertex to the indentation point by approximately straight ridges (three of the ridges are shown by red-dotted lines). By increasing the indentation, the triangular deformation mode becomes unstable as one of the vertices bifurcates into two new vertices that move away from each other leading to a square-like pattern. (B) Distribution of the elastic energy density in an elastic spherical cap with thickness in the range $0.0005 \leq t/R \leq 0.01$. Again, the critical indentations associated with transition between the axisymmetric and asymmetric deformation modes scale approximately linearly with the imposed indentation displacement. Interestingly, this critical indentation has approximately the same value for both rigid flat and point indentation experiments. On the other hand, the indentation associated with the appearance of four vertices, depends on the thickness with a weaker exponent. It should be emphasized here, again, that no imperfection was incorporated in our numerical simulations and for this reason our calculations provide an upper bound of the critical indentation associated with the transition between different asymmetric modes. The initial post-buckling response of spherical shells under point indentation has been predicted by the analytical model proposed by Fitch [40], which utilizes Marguerre’s non-linear shallow-shell theory [41]. This model predicts an initial post-buckling deformation of the shells as patterns consisting of three to five vertices depending on the spherical cap geometry, with indentation of shells with lower rise to thickness ratio (i.e. shallower and thicker shells) leading to asymmetric patterns with more vertices.

Despite the qualitative and quantitative similarities of the response of spherical caps under rigid plate and point-like indentations, there are also fundamental differences in their behaviors under these two indentation conditions. Under point-like indentation, the initial post-buckling shape of the spherical curvature condensate to the pyramid apex and the length of the corresponding spherical arc in the undeformed shape. This yields $Z^2 + 2(1 - Z)(1 - \sqrt{1 - r^2}) = (\sin^{-1}r)^2$, where $Z = Z/R$ is the normalized indentation and $r' = r/R$ is the normalized projected distance between vertices and the line of indentation. The numerical simulations shown in Fig. 4B qualitatively mimic the behavior observed in our experiments. The deformation map for the spherical shells with the center angle of $120^\circ$ is displayed in Fig. 4C, which was constructed by systematically varying the shell thickness in the range $0.0005 \leq t/R \leq 0.01$. The critical indentations associated with transition between the axisymmetric and asymmetric deformation modes scale approximately linearly with the imposed indentation displacement. This is further discussed in [39]. (C) Deformation map of a spherical cap subject to point indentation at its center. The dotted lines are best fit to the numerical results (filled circles and squares) in the form of $Z = C(t/R)^\beta$, where $C \approx 14$ and $\beta \approx 1$ for the fit to the filled circles and $C \approx 6.5$ and $\beta \approx 0.65$ for the fit to filled squares. The numerical simulations were carried out for $\alpha = 120^\circ$.

![Fig. 3. Deformation map of a spherical cap subject to flat rigid plate indentation. The dotted lines show the linear fit to the numerical results (filled circles and squares), $Z = C(t/R)$, where $C \approx -2.5$ and 13 for the two lines plotted. The numerical simulations were carried out for $\alpha = 120^\circ$.](image3)

![Fig. 4. (color online) Point indentation of a spherical cap. (A) Localization of deformation in a plastic spherical shell with $\tau \approx 0.35$ mm, $t/R \approx 100$ and $\alpha \approx 180^\circ$ indented at its center. After the initial axisymmetric deformation, the deformation localizes in the form of three vertices (denoted by yellow-dotted circles) that are connected together as well as the indentation point by approximately straight ridges (three of the ridges are shown by red-dotted lines). By increasing the indentation, the triangular deformation mode becomes unstable as one of the vertices bifurcates into two new vertices that move away from each other leading to a square-like pattern. (B) Distribution of the elastic energy density in an elastic spherical cap with $t/R \approx 0.005$ at two different normalized indentations, $Z = Z/R$ obtained using numerical simulations. The simulation shows that further increase in the indentation displacement leads to formation of additional vertices and ridges. This is further discussed in [39]. (C) Deformation map of a spherical cap subject to point indentation at its center. The dotted lines are best fit to the numerical results (filled circles and squares) in the form of $Z = C(t/R)^\beta$, where $C \approx 14$ and $\beta \approx 1$ for the fit to the filled circles and $C \approx 6.5$ and $\beta \approx 0.65$ for the fit to filled squares. The numerical simulations were carried out for $\alpha = 120^\circ$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image4)
shell is usually manifest in the form of three vertices, except for very shallow shells \( H/t < -30 \), where patterns comprising of four or five vertices appear. In contrast, when a spherical shell is subject to flat rigid plate indentation, in the range of geometrical parameters considered here, the initial post-buckling deformation mode typically has four or five vertices. The post-buckling behavior of the spherical cap under the two indentation conditions is also completely different. Under flat indentation, the initial asymmetric deformation mode persists over a large range of indentation, before the boundary effects become important as also seen in our qualitative experiments. However, under point indentation, after appearance of the asymmetric deformation mode, further increases in indentation make the system unstable and one of the vertices bifurcates into two new vertices that move away from each other, as shown in our qualitative experiments (try it yourself using a 1.5-l bottle). Numerical simulation of the systems indicates that this trend continues leading to formation of patterns with the number of vertices increasing with indentation. The formation of these new vertices in the shell is in general accompanied by a sudden reduction of the load-carrying capacity of the shell [39].

3. Pure bending of circular cylindrical shells

The response of cylindrical shells under pure bending has been the subject of much research during the last century. The pioneering work of [75] on elastic elbow ovalization under pure bending and its extension towards studying pipes and elbows (for example see: [42–44]) has motivated many further studies. Ovalization instability, which is one of the basic mechanisms of response, was introduced and studied by Brazier [45]. The role of material nonlinearity on the instability of circular shells under bending has been also studied in the last three decades [46–51]. An overview of the current state of the art on this subject is provided by Karamanos [52].

Here, we use detailed numerical simulations to explore the behavior and response of cylindrical shells of various thicknesses under pure bending. The emphasis is on qualitative aspects of the response and structural features that appear deep into the nonlinear regime, which is usually difficult to probe experimentally. Firstly, circular cylindrical shells with thickness \( t \), radius \( R \) and finite length \( L \) are considered under pure bending (Fig. 5A). Extension to oval cylindrical shells is discussed in Section 4.

![Fig. 5. (color online) Pure bending of a circular cylindrical shell.](image)

(A) Top: schematic of a circular cylindrical shell with radius \( R \), thickness \( t \) and length \( L \) under pure bending. Bottom: side view of a circular cylindrical shell deformed under pure bending. (B) Normalized moment-rotation response of two cylindrical shells. The unloading response is also shown for the cylindrical shell with \( t/R = 0.01 \) (dashed line). (C) Buckling shapes of cylindrical shells under pure bending. Buckled configurations of the cylindrical shells and the distribution of the elastic energy density in the region that the deformation localized in the form of a kink or multi-faceted patterns are shown for selected shell thicknesses. In all the calculations \( L/R = 10 \).
The two dimensionless geometric parameters that govern the response of the cylindrical shells are \( t/R \) and \( L/R \). The numerical model allows us to conduct a systematic parametric analysis of the role of these dimensionless parameters on the buckling and post-buckling response of elastic cylindrical shells. In this study, we focused on the role of \( t/R \), which is the more important of the two parameters, by considering both thin shells as well as thick shells, while keeping the length of the cylindrical shells constant, \( L/R = 10 \). As in the previous section, the computations were carried out for shells made of a linear elastic material with \( E = 100 \) GPa and \( v = 0.3 \). Four-node shell elements with reduced integration were employed and the automatic addition of volume-proportional damping was incorporated in the calculations, as discussed in the previous section. Furthermore, the cylindrical shells have no initial geometric or material imperfection.

The pure bending experiment of the cylindrical shells was numerically simulated by linearly increasing the rotation imposed to the end of the circular shell, while computing the reaction moment. Two rigid flat plates were attached to the cylinder ends. Pure bending experiment was simulated by rotating the two plates with respect to each other. With increasing applied rotation, the reaction moment drops drastically, Fig. 5B. The mechanism of deformation of cylindrical shells prior to bifurcation and localization of deformation is discussed by Kyriakides and Ju [48] and Ju and Kyriakides [49]. The buckling of the shell is due to the interplay of ovalization and bifurcation as discussed by Karamanos [52] and is imperfection sensitive in nature. Reissner [53] and Karamanos [52] have studied the role of the initial curvature of the beam on its response, which was shown to govern the interaction between the two instability modes (i.e. ovalization instability and bifurcation instability). By systematically varying the thickness of the cylindrical shells, two different buckling modes were observed. The critical stress for buckling of a perfect circular cylindrical shell under axial compression is \( \sigma_c = E t / \sqrt{3(1 - v^2)} \). Assuming that the buckling moment of the structure is the moment that leads the compressive stress at the outer extremity of the cylinder to reach this critical value, a simple calculation neglecting ovalization yields: \( M_c / RE_t^2 = 1.90 \), which is higher than the predictions from our numerical calculations. This simple calculation ignores the ovalization of the shell’s cross-section [76]. Short-wavelength ripples were predicted in our simulations prior to the bifurcation instability with wavelength \( \sim 3\sqrt{R} \), consistent with the experimental measurements on aluminum 6061-T6 cylindrical shells [48]. The unloading response of the shell with \( t/R = 0.01 \) is also depicted in Fig. 5B, which shows considerable hysteresis.

Fig. 5C shows the mapping between normalized thickness of a circular cylindrical shells with \( L/R = 10 \) and the buckling shapes. Buckling patterns and overall deformed configurations of the shells just after buckling are also shown for selected normalized thicknesses, \( t/R \). For \( t/R \geq 0.15 \), a single sharp kink or fold forms in the shell upon instability. In this regime, the ovalization mechanism of the cylindrical shell results in a reduction in the shell load-carrying capacity in the form of limit point instability. In this case, further increase of the applied rotation does not lead to the alteration of the buckling mode. The kink effectively works as a hinge in the system and the structure loses its load-carrying capacity after formation of the kink, as can be easily demonstrated by bending a straw. For thinner shells, \( t/R \leq 0.15 \), multi-faceted, diamond-like buckling shapes appear with a characteristic length, which diminishes for very thin shells. This buckling mode consists of vertices with a very high elastic energy density, which are connected by approximately straight ridges. The number of vertices, as well as the characteristic length of the ridges depends on the normalized shell thickness. Formation of these multi-faceted patterns in thin shells has been observed in many experimental studies. The post-buckling deformation of these shell structures is accompanied by instabilities which cause the formation of additional ridges and vertices with sudden reduction in the load-carrying capacity of the structure.

4. Pure bending of oval cylindrical shells

In this section, we have extended our study to the case of oval cylindrical shells, Fig. 6A. This work complements the earlier theoretical and experimental work on buckling of oval cylindrical shells under various loading conditions [54–58]. Fig. 6B displays the moment-rotation responses of oval cylindrical shells with various ellipticity \( e \), defined as the ratio of the major-to-minor axes of the cylinder cross-section, denoted by \( b \) and \( a \), respectively (i.e. \( e = b/a \)). The normalized thickness of the shells is \( t/a = 0.01 \) in this set of calculations. The calculations were performed by keeping the minor axis constant, so that the cross-sectional area moment of inertia, \( I = (\pi/4)a^2(3e^2+1) \), increases linearly upon increasing the ellipticity of the cylindrical shell. This leads to a stiffer response as the eccentricity increases before the onset of instability. As a first approximation, the critical stress for buckling of an oval cylindrical shell under axial compression can be
estimated from $\sigma_c = \frac{E}{\sqrt{2B}} \sqrt{3(1-v^2)}$. Assuming that the buckling moment of the structure is the moment that causes the compressive stress at the outer extremity of the cylinder to reach the critical value, simple calculation yields: $M_c = \frac{\pi}{4\sqrt{3(1-v^2)}} (3e + 1)^e$, which predicts the decrease in the critical moment upon increasing the eccentricity of the shell, as also seen in our numerical simulations, Fig. 6B. Here, the addition of the material (by increasing the ellipticity, keeping the minor axis constant) leads to a stiffer but weaker shell. No attempt was made to quantify the role of ovalization on the reduction of the buckling load other than the limited calculations presented in Fig. 6B.

We also studied the post-buckling response of oval cylindrical shells using numerical simulations. For cylindrical shells with high eccentricity ($e = 4$ in Fig. 6B), the post-buckling deformation of the shell leads to a higher load-carrying capacity than the bifurcation load. Similar behavior was predicted for the response of oval cylindrical shells under pure compression both theoretically and experimentally [54,55]. By systematically varying the thickness and eccentricity of oval shell structures, we have developed a deformation map for these structures, which characterizes their post-buckling behavior, in Fig. 7A. Similar to the buckling of circular cylindrical shells, two buckling modes were observed in the range of geometrical parameters considered in this study. Again thin shell buckling is characterized by multifaceted patterns, while in thicker shells a single-fold (sharp kink) appears. The critical thickness associated with the transition between these two buckling shapes, slightly increases by increasing the shell eccentricity. As discussed for circular cylindrical shells under pure bending by Kyrkides and Ju [48] and Ju and Kyrkides [49], short-wavelength ripples may appear on the compressed side of the shell, prior to localization of the deformation. For thick finite oval cylindrical shells, the appearance of these short-wavelength ripples may cause instability and sudden decrease in the load-carrying capacity of the shell structure, Fig. 7B. Further increase in the imposed rotation, makes these ripples unstable and leads to the formation of a single kink. This phenomenon was not observed for circular shells in the range of geometrical parameters considered in this study, where the structure retains its load-carrying capacity as the short-wavelength ripples appear prior to the bifurcation instability.

As discussed briefly above, shells with high eccentricity do not lose much of their load-carrying capacity, which is in contrast with the circular shells, where buckling is catastrophic most of the time. When oval cylindrical shells are deformed highly into the post-buckling regime, the core of the initial fold can become unstable and locally buckle as shown in Fig. 8. This transition is catastrophic and the shells lose most of their load-carrying capacity.

5. Discussions and concluding remarks

The inextensible or almost inextensible deformation of elastic shells under mechanical loading leads to the formation of intricate structural features at a much smaller scale than the overall structure size. These features, which can be seen by simple experiments in everyday life as well as in biological and engineering systems, are associated with high-energy density and evolve in intricate ways as the shell is further loaded deep into the nonlinear regime. The qualitative response of thin shell structures is crucially controlled by the natural curvature of the surface, as discussed by Vaziri and Mahadevan. [39] using a linearized theory for shell deformation as well as detailed numerical simulations of elastic shells with negative, zero and positive Gaussian curvature. The geometrically nonlinear character of the post-buckling deformation of shells precludes the use of purely analytical techniques to understand them, while also posing fundamental challenges for experimentation on shells. Here, we studied the deformation and response of two basic shell structures—cylinders under pure bending and a segment of sphere (cap) under point-like and flat rigid indentation. For indentation of elastic caps, numerical simulations and qualitative experiments were carried out to unravel some of the physical mechanisms of the response, including various stages of
deformation that lead to localization of deformation in the form of vertices and ridges. For pure bending of cylindrical shells, we showed that bifurcation instability of cylindrical shells leads to two distinct buckling modes—single fold and multi-faceted pattern—depending mainly on the shell thickness. For oval cylindrical shells, the shell eccentricity governs the post-buckling behavior of cylindrical shells both in the initial linear regime and in the post-buckling regime.

When studying the mechanics of shell structures, another dimension of complexity arises when considering multi-walled structures such as multi-walled nanotubes and microcapsules [59–62]. In these structures, the bending and stretching stiffness could scale differently with the number of layers depending on the interface condition and the interactions between different layers (e.g. van der Walls forces, friction, etc.), which in turn may lead to the emergence of peculiar structural features and the new phenomena [3]. Moreover, individual layers could buckle and alter the interaction condition between different layers, influencing the overall response and mechanics of these structures. An example is a periodic rippling manifest at both macro- and micro-scales in multi-walled circular shell structures under bending. Fig. 1B, which was not observed in single-walled circular shells. It is interesting to note that this periodic rippling was observed in the simulations of the bending response of a finite oval cylindrical shell. Another interesting example is the faceted shape of multi-walled nanoparticles which form as they adhere to substrates [63]. Formation of the faceted pattern is related to the defect-mediated change from coherent-to-incoherent bending, where slip occurs between neighboring layers [64].

Finally, the response and, more specifically, the buckling of shell structures is known to be imperfection sensitive. Under axial compression, geometrical imperfection can lead to a factor of up to 5 difference between the experimentally measured buckling loads and the theoretical values [65–69]. Other types of imperfection such as crack and inclusions can also alter the response significantly [70–72]. The role of imperfections on the mechanics of highly deformed elastic shells and on localization of deformation and subsequent appearance of structural features such as vertices and ridges studied here is not well-understood.

In conclusion, despite immense efforts expended in the study of the mechanics of highly deformed shells, our understanding of many aspects of shell behavior is still at its infancy and calling for rigorous studies from the theoretical, numerical and experimental point of view.

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